Department of Mathematics and Statistics
Lake shore campus | 1032 W. Sheridan Road | Chicago, Illinois 60660
Phone 773.508.3558 | Fax 773.508.2123 | www.math.luc.edu

Math 131 - Fall 2022 - Final Exam

Key

Print name:____

Section number:__________ **Instructor's name:**______________________________

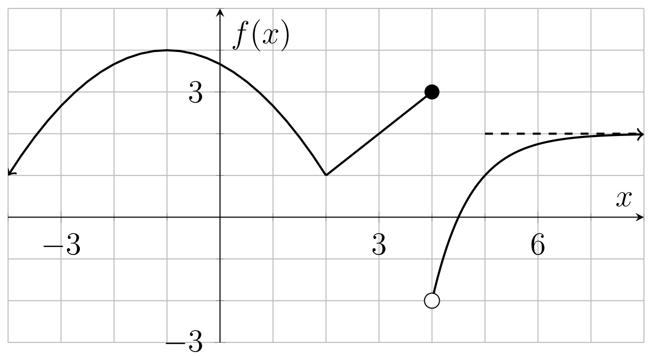
Directions:

- This exam has 11 questions.
- You will have two hours to complete this exam.
- It will be graded out of 100 points.
- Show your work. Answers (even correct ones) without the corresponding work will receive no credit.
- A formula sheet has been provided on the last page of this exam. You may not refer to any other notes or textbooks during the exam.
- You may use a calculator as long as it is unable to connect to the internet.
- You may only speak with your instructor during the exam.

Good luck!

Question:	1	2	3	4	5	6	7	8	9	10	11	Total
Points:	12	6	10	6	10	12	10	11	10	5	8	100
Score:												

1. [12 points] Using the graph below, find each of the following. If the answer does not exist, write "DNE". The dotted line on the right of the graph represents a horizontal asymptopte and is not a part of the function.



(a) [2 points] $f(2)$

(b) [2 points] $\lim_{x \to -1} f(x)$

(c) [2 points] $\lim_{x \to \infty} f(x)$

(d) [2 points] $\lim_{x \to 4^+} f(x)$

(e) [2 points] $\lim_{x \to 4^-} f(x)$

(f) [2 points] $\lim_{x \to 4} f(x)$

Solution:

(a) $f(2) = 1$

(b) $\lim_{x \to -1} f(x) = 4$

(c) $\lim_{x \to \infty} f(x) = 2$

(d) $\lim_{x \to 4^+} f(x) = -2$

(e) $\lim_{x \to 4^-} f(x) = 3$

(f) $\lim_{x \to 4} f(x) = DNE$

- 2 points each

2. [6 points] Suppose $f(x) = x^2 - 5x + 1$.

(a) [3 points] Write the limit definition of $f'(4)$.

Solution: $f'(4) = \lim_{h \to 0} \frac{f(4+h) - f(4)}{h}$ OR $f'(4) = \lim_{x \to 4} \frac{f(x) - f(4)}{x - 4}$

- 1.5 point for h goes to 0 OR x goes to 4
- 1.5 point for difference quotient

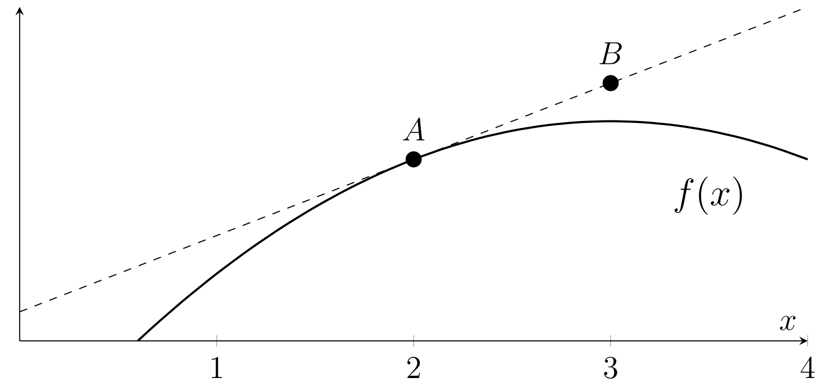
(b) [3 points] Evaluate the limit you found in part (a). *Do not use the power rule or other shortcuts!*

Solution:

$$f'(4) = \lim_{h \to 0} \frac{f(4+h) - f(4)}{h}$$
$$f'(4) = \lim_{h \to 0} \frac{(4+h)^2 - 5(4+h) + 1 - (4^2 - 5 \cdot 4 + 1)}{h}$$
$$f'(4) = \lim_{h \to 0} \frac{(16 + 8h + h^2) - 20 - 5h + 1 - (-3)}{h}$$
$$f'(4) = \lim_{h \to 0} \frac{3h + h^2}{h}$$
$$f'(4) = \lim_{h \to 0} 3 + h = \boxed{3}$$

- 1 point for substitution step
- 1 point for expansion
- 1 point for simplification and answer

3. [10 points] The function in the figure has $f(2) = 5$ and $f'(2) = 2.1$.



(a) [4 points] Find the formula for the tangent line to $f(x)$ at $x = 2$.

Solution: $y = 5 + 2.1(x - 2)$ OR $y = 2.1x + 0.8$

- 1 point for finding correct slope
- 1 point for identify the pair $(x, f(x)) = (2, 5)$
- 2 points for assembling the tangent line formula

(b) [3 points] Use the picture and your equation from part (a) to find the coordinates for point B. Present your answer as an (x, y) pair.

Solution: If $x = 3$ then $y = 7.1$. $B = (3, 7.1)$

- 2 points for substitution into part a)
- 1 point for writing in pair (x, y)

(c) [3 points] What is the slope of the tangent line for $h(x) = x \cdot f(x)$ at $x = 2$?

Solution: We have $h'(x) = f(x) + xf'(x)$ so $h'(2) = 5 + 2(2.1) = 9.2$.

- 2 points for taking derivative
- 1 point for substitution

4. [6 points] A sports car goes from 0 mph to 60 mph in five seconds. Its velocity is given in the following table, converted from miles per hour to feet per second, so that all time measurements are in seconds.

Time, t (sec)	0	1	2	3	4	5
Velocity, $v(t)$ (ft/sec)	0	30	52	68	80	88

(a) [2 points] The **position** function is (circle one)

concave up **concave down** **constant**

Solution: The position function is **concave up** because the velocity is increasing.

- 2 points for the right answer

  Note: Students can get partial points if they explain correctly but circle the wrong answer. However, no need to have an explanation.

(b) [4 points] Find the average **acceleration** between $t = 0$ and $t = 2$.

Solution:

$$\frac{v(2) - v(0)}{2 - 0} = \frac{52}{2} = 26 \text{ ft/sec}^2.$$

- 2 points for the setting up the quotient
- 2 point for substitution and correct answer

5. [10 points] Suppose $f(x) = \ln(\cos(x)) + 2^x$.

(a) [5 points] Find $f'(x)$.

Solution:

$$f'(x) = \frac{1}{\cos(x)}(\cos(x))' + \ln(2)2^x$$

(1 point for derivative of ln , 1 point for derivative of 2^x, 2 points for chain rule).

$$= \frac{-\sin(x)}{\cos(x)} + \ln(2)2^x \quad \text{(1 point)}$$

$$= -\tan(x) + \ln(2)2^x$$

Note: If an instructor wishes to grade for simplification in the last step, adjust the points to 3 points, 1 point, 1 point.

(b) [5 points] Find $f''(x)$. *You do not need to simplify your final answer.*

Solution:

$$f''(x) = \left(\frac{-\sin(x)}{\cos(x)}\right)' + (\ln(2)2^x)' \qquad \text{OR} \quad (-\tan(x))' + (\ln(2)2^x)' \quad \text{(2 point)}$$

$$= -\frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} + \ln(2)\ln(2)2^x \qquad \text{OR} \quad \frac{-1}{\cos^2(x)} + \ln(2)\ln(2)2^x \quad \text{(3 points)}$$

$$= \frac{-1}{\cos^2(x)} + (\ln(2))^2 2^x \qquad \text{OR} \quad -\sec^2(x) + (\ln(2))^2 2^x$$

6. [12 points] The gross domestic product (GDP) of a certain country during the recovery from a national crisis (at $t = 0$) is approximated by

$$G(t) = -0.4t^3 + 4.8t^2 + 20 \text{ for } 0 \leq t \leq 12,$$

where $G(t)$ is measured in billions of dollars and t is measured in years.

(a) [4 points] Find a formula for $G'(t)$ and interpret the expression $G'(3)$ in the context of this question. *Make sure to write your answer in a complete sentence with units.*

Solution: We have $G'(t) = -1.2t^2 + 9.6t$ so $G'(3) = 18$. This means that 3 years into the recovery the GDP is increasing by \$18 billion per year.

- 1 point for taking derivative
- 1 point for substitution
- 2 points for interpretation with units

(b) [4 points] Find the critical points of $G(t)$ and classify them as minimums, maximums, or neither.

Solution: We know $G'(t) = -1.2t(t - 8) = 0$ when $t = 0$ or $t = 8$. The second derivative is $G''(t) = -2.4t + 9.6$. We have that $G''(0)$ is positive and $G''(8)$ is negative so $t = 0$ is a minimum and $t = 8$ is a maximum. Or, we can use the first derivative test to classify them

- 2 points for setting up equation $G'(t) = 0$, and solving for $t = 0$ and $t = 8$
- 2 point for using either first or second derivative test.

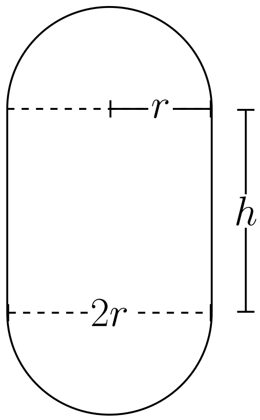
(c) [4 points] Identify any inflection points for this function on its domain.

Solution: Note that $G''(t) = -2.4t + 9.6$ changes sign at $t = 4$ so this is an inflection point for $G(x)$.

- 1 point for setting up equation $G''(t) = 0$, either implicitly or explicitly
- 1 point for correctly taking derivative to get $G''(t)$
- 1 point for solving for $t = 4$
- 1 point for explaining the signs change

7. [10 points] A window has the shape of a rectangle with a semi circle on top and bottom; the diameter of the semicircles exactly match the width of the rectangle.

(a) [5 points] Write an equation for the area of the window in terms of the radius r of the window's circular part given that the window has a perimeter of 10π inches.(Hint. The area of a circle is πr^2 and the circumference of a circle is $2\pi r$.)



Solution: 2 points

$$A = \pi r^2 + 2r \cdot h$$

3 points

Using $P = 2\pi \cdot r + 2h = 10 \cdot \pi$ we can find h in terms of r and express the area as a function of radius.

$$h = 5\pi - \pi r$$

$$A(r) = \pi r^2 + 2r \cdot (5\pi - \pi r) = 10\pi r - \pi r^2$$

(b) [5 points] What is the maximum area the window can cover?

Solution: 3 points

$$A'(r) = 10\pi - 2\pi r$$

Finding critical point

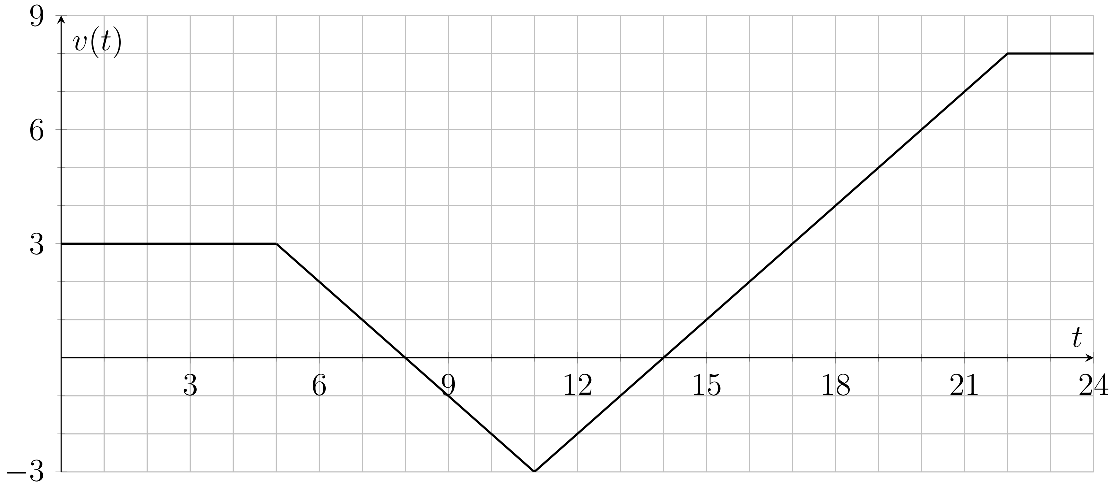
$$A'(r) = 10\pi - 2\pi r = 0 => r = 5$$

2 points

Using second order derivative $A''(r) = -2\pi < 0$ (concave down), $r = 5$ is a point of local and global maximum, so the maximum area is given by

$$A(5) = 10\pi \cdot 5 - \pi \cdot 5^2 = 25\pi = 78.5398...$$

8. [11 points] A ship is moving out to sea directly away from (and sometimes directly towards) the shore. This graph shows the velocity for the ship over a 24 hour period of time. Time is measured in hours and the velocity is measured in km per hour.



(a) [5 points] Complete the table for the distance the ship is from shore after t hours.

Time (hrs)	0	5	11	14	22	24
Distance (km)	0					

Solution:

Time (hrs)	0	5	11	14	22	24
Distance (km)	0	15	15	10.5	42.5	58.5

- 1 point for each correct answer

(b) [2 points] What is the **average value** of the velocity of the ship over the first 14 hours of the trip?

Solution: 2 points Average velocity over the first 14 hours $= \frac{1}{14-0} \cdot \int_0^{14} v(t)dt = 0.75$ OR $= \frac{10.5-0}{14-0} = 0.75$

(c) [2 points] On what time interval(s) is the distance from the shore decreasing?

Solution: 2 points $8 < t < 14$ since $v(t) < 0$ on this interval.

(d) [2 points] In the **first 12 hours** of the trip when is the ship a maximum distance from the shore?

Solution: 2 points The ship is a maximum distance from the shore at 8 hours.

9. [10 points] (a) [5 points] Compute the indefinite integral $\int \left(7 - 3e^x + \frac{4}{x}\right) dx$

Solution: 1 point for breaking it down as a sum of antiderivatives

$$\int (7 - 3e^x + \frac{4}{x})dx = \int 7dx - \int 3e^x dx + \int \frac{4}{x}dx =$$

1 point for each correct antiderivative, 1 point for the constant of integration

$$\int 7dx - \int 3e^x dx + \int \frac{4}{x}dx = 7x - 3e^x + 4\ln(|x|) + C$$

(b) [5 points] Find the antiderivative $F(x)$ for $f(x) = x^4 + 6x - \sin(x)$ that satisfies the property $F(0) = 3$.

Solution: 3 points for finding correctly general antiderivative

$$F(x) = \int (x^4 + 6x - \sin x)dx = \frac{x^5}{5} + 3 \cdot x^2 + \cos(x) + C$$

2 points for finding correctly the value of the constant

$$F(0) = \frac{0^5}{5} + 3 \cdot 0^2 + \cos(0) + C = 3$$

$$1 + C = 3 => C = 2$$

$$F(x) = \frac{x^5}{5} + 3x^2 + \cos(x) + 2$$

10. [5 points] Find the value of $\int_0^6 (f(x) + 3 \cdot g(x))\, dx$ given $\int_6^0 f(x)\, dx = 5$, and $\int_0^6 g(x)\, dx = 2$.

Solution: 1 point

$$\int_0^6 (f(x) + 3g(x))dx = \int_0^6 f(x)dx + \int_0^6 3g(x)dx =$$

1 point

$$\int_0^6 f(x)dx + \int_0^6 3g(x)dx = \int_0^6 f(x)dx + 3\int_0^6 g(x)dx =$$

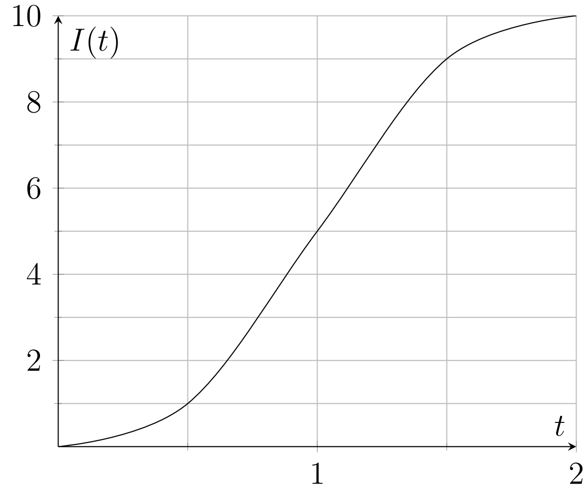
2 points

$$\int_0^6 f(x)dx + 3\int_0^6 g(x)dx = -\int_6^0 f(x)dx + 3\int_0^6 g(x)dx =$$

1 point

$$-\int_6^0 f(x)dx + 3\int_0^6 g(x)dx = -5 + 3 \cdot 2 = 1$$

11. [8 points] The figure below shows the amount of electrical current $I(t)$, measured in coulombs per second, flowing through a wire t seconds after a switch is flipped.



(a) [5 points] Estimate $\int_0^2 I(t)\,dt$ using a right Riemann sum with 4 subdivisions.

Solution:

- 4 points

$$\int_0^2 I(t)dt \approx 1 \cdot 0.5 + 5 \cdot 0.5 + 9 \cdot 0.5 + 10 \cdot 0.5 =$$

- 1 points

$$= (1 + 5 + 9 + 10) \cdot 0.5 = 12.5 \text{ coulumbs}$$

(b) [3 points] Interpret $\int_0^2 I(t)\,dt$ in the context of this question. *Make sure to write your answer in a complete sentence with units.*

Solution: 3 points The definite integral

$$\int_0^2 I(t)dt$$

in coulombs gives a total amount of the electrical current flowing through a wire during the first two seconds.

Five derivative rules for operations on functions.

Constant Multiple Rule: $\frac{d}{dx}[cf(x)] = cf'(x)$

Sum and Difference Rule: $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$

Product Rule: $\frac{d}{dx}[f(x) \cdot g(x)] = f'(x)g(x) + f(x)g'(x)$

Quotient Rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

Chain Rule: $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

Ten derivative rules for functions

Derivative of a Constant: $\frac{d}{dx}[c] = 0$, where c is a constant.

The Power Rule: $\frac{d}{dx}[x^n] = nx^{n-1}$

Exponential Functions: General Case: $\frac{d}{dx}[a^x] = a^x \cdot \ln(a)$

Exponential Functions: Special Case: $\frac{d}{dx}[e^x] = e^x$

Three Trigonometric Rules. $\frac{d}{dx}[\sin(x)] = \cos(x)$

$\frac{d}{dx}[\cos(x)] = -\sin(x)$

$\frac{d}{dx}[\tan(x)] = \sec^2(x) = \frac{1}{\cos^2(x)}$

Three Inverse Function Rules

$\frac{d}{dx}[\ln(x)] = \frac{1}{x}$

$\frac{d}{dx}[\arctan(x)] = \frac{1}{1+x^2}$

$\frac{d}{dx}[\arcsin(x)] = \frac{1}{\sqrt{1-x^2}}$

General Antiderivative Rules

If k is a constant $\int k\,dx = kx + C$

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$, when $n \neq -1$

$\int a^x\,dx = \frac{a^x}{\ln(a)} + C$

$\int e^x\,dx = e^x + C$

$\int \cos(x)\,dx = \sin(x) + C$

$\int \sin(x)\,dx = -\cos(x) + C$

$\int \sec^2(x)\,dx = \tan(x) + C$

$\int \frac{1}{x}\,dx = \ln(|x|) + C$

$\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$

$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C$